Where combinatorics meets Fundamental Physics

Rahul K Balaji

Physics 502
Princeton University

April. 17, 2025

Motivation

Some Background

- $\mathcal{A}_{m\to n} = \langle q_1^{r_1}, \ldots, q_n^{r_n} | p_1^{r_1}, \ldots, p_m^{s_m} \rangle$: related to the probability of an event occuring.
- The amplitudes factorize on physical poles: $X_{i,j} = (k_i + \ldots + k_{j-1})^2 = m^2$ using unitarity and locality
- Feynman diagrams introduce spurious poles and physical poles are manifest over multiple channels

Punchline

- Factorization of n-point amplitudes in lower point amplitudes using the **kinematic mesh**

$$\mathcal{A}_n(c_* \neq 0) = \left(\frac{1}{X_B} + \frac{1}{X_T}\right) \times \mathcal{A}^{down} \times \mathcal{A}^{up}$$

- You can see the same patterns of zeros and an avatar of the factorization near zeros generalizes to all interesting theories of colored particles: the Non-linear Sigma Model (NLSM) for pions, as well as gluons in Yang-Mills theory (YM)

Why $\mathrm{Tr}\phi^3$

$$\mathcal{L} = \partial_\mu \phi^a \partial^\mu \phi^a + g\mathrm{Tr}\phi^3$$

$$\mathrm{Tr}\phi^3 = f^{abc}\phi^a\phi^b\phi^c ; \phi(x) = \phi^a(x)T^a$$

- Simplest non-Abelian colour gauge theory - no spins or gauge fixing
- One to One correspondence between Feynman diagrams and rooted trees - makes manifest the **combinatorics**
- Connections to Matrix models and BCJ relations

Outline

1. Motivation
2. Definitions and Setup
3. Examples
4. Back to the Punchline

Definitions and Setup

The Momentum Polygon

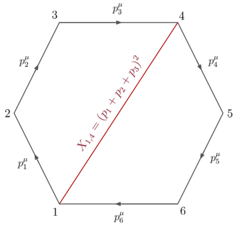


Figure 1: 6-point momentum polygon.

- $X_{i,j} = (p_1 + \ldots + p_{j-1})^2$ are naturally associated with propagators that appear in Feynman diagrams, and $\mathcal{A}_n^{\mathrm{Tr}\phi^3} \equiv \mathcal{A}_n^{\mathrm{Tr}\phi^3}(\{X_{i,j}\})$
  - $X_{i,i+1} = p_i^2 = 0$.
  - Good basis as it respects cyclical symmetry and momentum conservation
- $c_{i,j} = -2p_i \cdot p_j = X_{i,j} + X_{i+1,j+1} + X_{i,j+1} + X_{i+1,j}$

The Kinematic mesh

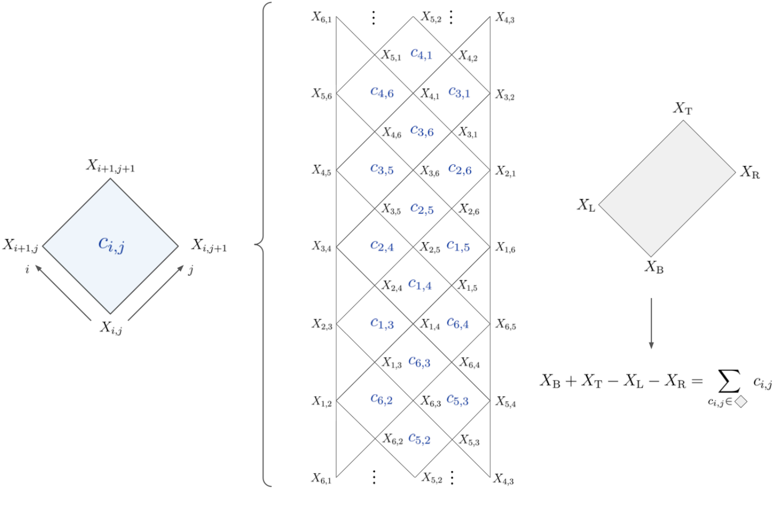
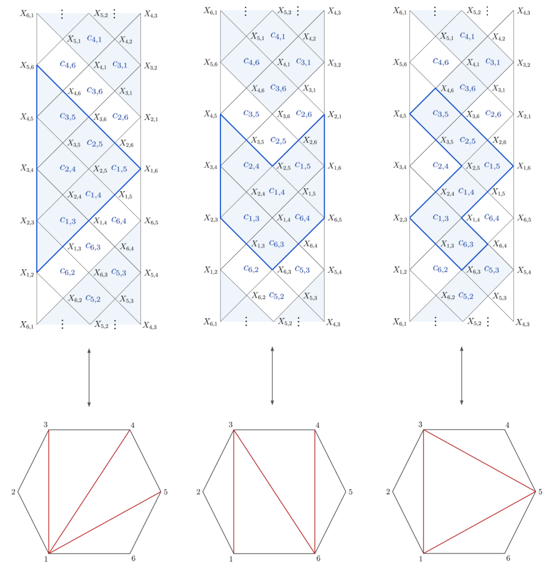


Figure 2: Kinematic mesh for 6-point.

The Kinematic mesh and the momentum polygon



The ABHY Associahedron

- For an n-point amplitude this is an $(n-3)$-dimensional polytope
- Define region in kinematic space$\Delta_+ = \{X_{i,j} > 0, \mathrm{for all} i < j \in (1, \ldots, n)\}$
- Pick a subregion of the mesh determining a basis of $(n-3)$ planar variables $\tilde{X}_{i,j}$ and $(n-2)(n-3)/2$ $\tilde{c}_{i,j}$s
- Impose $\tilde{c}_{i,j} > 0$. Δ_+ defines $n(n-3)/2$ inequalities in an $(n-3)$ dimensional space spanned by $\tilde{X}_{i,j}$. The convex hull is the ABHY associahedron.

Outline

1 Motivation

2 Definitions and Setup

3 Examples

4 Back to the Punchline

Examples

4-point Amplitude (Warm up)

$$\mathcal{A}_4 = \frac{1}{X_{1,3}} + \frac{1}{X_{2,4}}$$

- $X_{1,3}$ and $X_{2,4}$ correspond to the Mandlestans s and t. These are in turn related to the non-planar variable $c_{1,3} = u$

$$X_{1,3} + X_{2,4} = c_{1,3}$$

- So picking the basis $\{X_{1,3}, c_{1,3}\}$ the ABHY associahedron is

$$\{X_{1,3} > 0 \wedge X_{2,4} > 0 \iff X_{1,3} < c_{1,3}\} \iff 0 < X_{1,3} < c_{1,3}$$

- Which is simply a line segment, or a **one-simplex**

5-point Amplitude

- The Amplitude:

$$\mathcal{A}_5 = \frac{1}{X_{1,3}X_{1,4}} + \frac{1}{X_{2,4}X_{2,5}} + \frac{1}{X_{1,3}X_{3,5}} + \frac{1}{X_{1,4}X_{2,4}} + \frac{1}{X_{2,5}X_{3,5}}$$

- The Associahedron:

$$\begin{cases} X_{1,3} > 0 \\ X_{1,4} > 0 \\ X_{2,4} > 0 \iff c_{1,3} - X_{1,3} + X_{1,4} > 0 \\ X_{2,5} > 0 \iff c_{1,3} + c_{1,4} - X_{1,3} > 0 \\ X_{3,5} > 0 \iff c_{1,4} + c_{2,5} - X_{1,4} > 0 \end{cases}$$

The Associahedron and Minkowski sums of Simplices

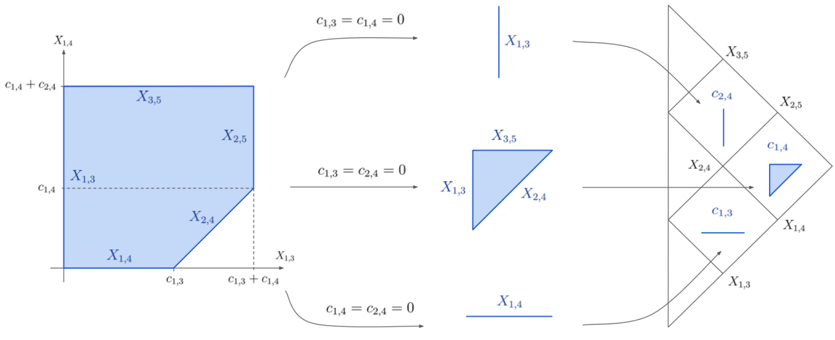


Figure 4: 5-point ABHY associahedron and respective Minkowski summands.

Connecting back to the Amplitude

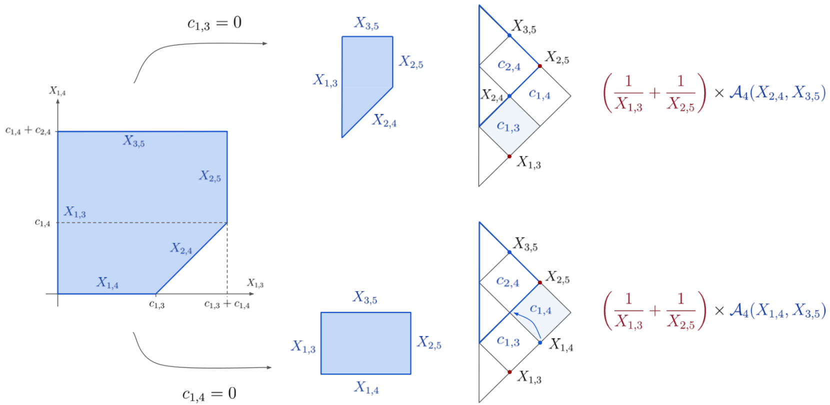


Figure 5: 5-point factorization near zeros.

Factorization in the 5-point case

$$\mathcal{A}_5(X_{1,3}, X_{1,4}, X_{2,4}, X_{2,5}, X_{3,5}) \xrightarrow{c_{1,4}=0} \left(\frac{c_{1,3}}{X_{1,3}X_{2,5}}\right) \times \left(\frac{1}{X_{1,4}} + \frac{1}{X_{3,5}}\right)$$

$$\mathcal{A}_5(X_{1,3}, X_{1,4}, X_{2,4}, X_{2,5}, X_{3,5}) \xrightarrow{c_{1,3}=0} \left(\frac{c_{1,4}}{X_{1,3}X_{2,5}}\right) \times \left(\frac{1}{X_{2,4}} + \frac{1}{X_{3,5}}\right)$$

Outline

1 Motivation

2 Definitions and Setup

3 Examples

4 Back to the Punchline

Back to the Punchline

Back to the Punchline

$$\mathcal{A}_n(c_* \neq 0) = \left(\frac{1}{X_B} + \frac{1}{X_T}\right) \times \mathcal{A}^{down} \times \mathcal{A}^{up}$$

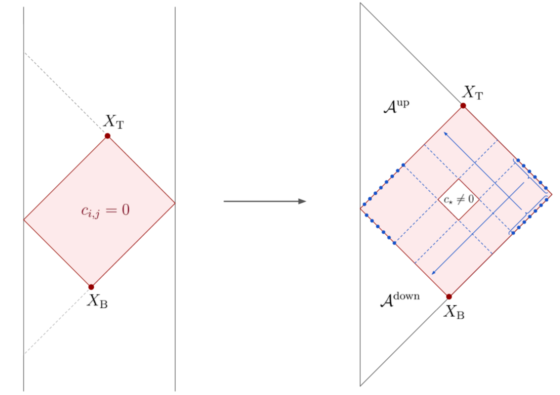


Figure 8: Zeros (left) and Factorizations with associated kinematic shifts (right).

Questions!

1 Motivation

2 Definitions and Setup

3 Examples

4 Back to the Punchline

Appendix

A1