Cutting Open Matrix Models

Pre Thesis

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Motivation

Some Background

- Random matrix theory, originally developed to study statistics of eigenvalues in nuclear physics [Wigner, Wishart et al. '50-60's]
- Hermitian matrix models as a "scalar field theory in zero dimensions" ['t Hooft '74]

$$Z = \int dM \exp\left(\frac{N}{g_s}\mathrm{Tr}V(M)\right)$$

- 't Hooft double line notation → Feynman Diagrams (Ribbon Graphs/Fatgraphs)

First Steps towards Combinatorics in Matrix Models

- Planar Free Energy is the generating function of planar maps with vertices of prescribed valencies [BIPZ '78]
- Z are tau-functions of integrable hierarchies, and satisfy differential constraints of the form of Virasoro Algebra.[Douglas, Shenker, Gross, Migdal, Brezin Kazakov, etc.]
- Purely combinatorially, recursion in the form of *Tutte equations*. Describes cutting and contracting edges of grapgs, and gluing boundaries.

Combinatorics of Amplitudes

- BCJ Relations, BCFW recursion etc.
- Positive geometry methods (Amplituhedron, Associahedron etc.) [Arkani-Hamed et al.]
- *Surface Functions* and the *Cut Equation* [Arkani-Hamed, Frost, Salvatori '24]

Overview: Deforming Away from the Matrix Model

(Why?)Toy Models:

- Surface functions G_S: Topological surfaces S with boundary, that retain a lot of the combinatorics of colored scalar amplitudes, and gauge theories (NLSM and YM)
- Stringy Integrals $G_S^{\alpha'}$: Integrals over open string moduli spaces that are similar to open-string amplitudes (but much simpler)

Overview

Bridging the combinatorics of Matrix models and surface functions using Virasoro constraints and the cut equation is mahtematically interesting because of the Lie Algebra perspective to the *Mapping Class Group Action*

Outline

Surfaceology

Surface functions

- G_S are generating functions of topologically inequivalent *polyangulations* of the topological surface S
- They are polynomials in finitely many variables x_C associated to the (mapping-class-group orbits of) paths on S with endpoints on the boundary of S
- These x_C variables can be thought of as "inverse propagators" of a QFT integrand, but where we "forget" some information about loop redefinitions.

Surface functions in use

- If S is a disk with marked points on the boundary, labelled $1, 2, \ldots, n$, G_S is the tree amplitude for a colored scalar Φ_I^J with some scalar interaction lagrangian

$$\mathcal{L} = \sum_{k=3} \frac{t_k}{k} \mathrm{Tr}\Phi^k.$$

- At four points, we write this tree amplitude as

$$G_{1,2,3,4} = x_{13}t_3^2 + x_{24}t_3^2 + t_4$$

Surface functions in use

- $S = \circledcirc$, the annulus with a marked point on each boundary, labelled 1 and 2. There is only one distinct triangulation of S, which is given by two curves from 1 to 2, so that

$$G_{\circledcirc} = x_{12}^2 t_3^2 + x_{12} t_4,$$

- A more complicated example is $S =$ , the torus with a hole and no marked points. In this case

G  $= \frac{1}{3} x^3 t_3^2 + \frac{1}{2} x^2 t_4.$

Matrix Model Limit

- in the limit that all x-variables are set equal, $x_C = x$, G_S resembles a contribution to a matrix model free energy. For example, in the case of a cubic Lagrangian,

$$G_S(x, t_3) = F_S\, x^E t_3^V\,,$$

- F_S is the number of cubic fatgraphs (up to symmetry factors) that can be drawn on the surface S, which all have $E = n - 3 + 3L$ edges and $V = 2n - 2 + 2L$ vertices.The loop order, L, is given by $L = 2g + b - 1$ where S has genus g and b boundary components.

Matrix Model Limit

$F = -\log Z$ of the Hermitean matrix model with cubic potential,

$$Z = \int dM e^{-\frac{1}{\hbar}\operatorname{Tr} V(M)}, \qquad V(M) = \frac{1}{2x}M^2 + \frac{t_3}{3}M^3.$$

In particular

$$F = 1 + \sum_{L=2} \hbar^{L-1} \sum_{2g+b=L+1} N^b G_{S_{g,b}}(x, t_3),$$

where $S_{g,b}$ is the genus g surface with b boundaries and no marked points.

Outline

The Cut Equation

Cut Equation in Action

- Before taking this "matrix model limit", the surface functions G_S satisfy a remarkably simple constraint:

$$\partial_{x_C} G_S = G_{S\setminus C},$$

- For example, cutting the torus with one hole open along its A-cycle (say) gives an annulus, and indeed:

$\partial_x G$  $= x^2 t_3^2 + x t_4 = G_{\circledcirc}.$

- Moreover, cutting the annulus (with a marked point on each boundary) along the curve connecting its two boundaries gives a 4-point disk:

$$\partial_{x_{12}} G_{\circledcirc} = x_{12} t_3^2 + x_{12} t_3^2 + t_4 = G_{1,2,1,2}.$$

Putting cuts in context

- *Hepp bound* of the partial amplitudes in a colored scalar theory.
- every vacuum contribution G_S (for S with no marked points) can be recursively solved in terms of the tree-level amplitudes $G_{1,\ldots,n}$. In other words, every vacuum surface S can be cut open into a product of disks.
- By contrast, the well known constraints on the matrix model free energy F — the *Virasoro constraints* or *Schwinger-Dyson equations* — are constraints only on the numbers of *vacuum* fatgraphs.

The Bridge

How are these combinatorial constraints on the numbers of vacuum graphs related to the combinatorial constraints imposed by the cut equation?

Outline

Matrix Models

Review

- The 1-Hermitian Matrix Model

$$Z = \int dM\, e^{-\frac{1}{\hbar}\operatorname{Tr} V(M)}, \qquad V(M) = \sum_k \frac{t_k}{k} M^k.$$

- The Full series can be written out as:

$$F = \sum_{\{V_k\}\geq 0} \prod_{k\geq 3} \left(\frac{-t_k}{k\hbar}\right)^{V_k} \left\langle\!\!\left\langle \frac{1}{\prod_k V_k!} \prod_{k\geq 3} \operatorname{Tr}\left(M^k\right)^{V_k} \right\rangle\!\!\right\rangle_c ,$$

So grouping terms together we get

$$F = -\log Z = \sum_{g,p\geq 0} \hbar^{L-1} N^p G_{g,p}(t),$$

$$G_{g,p}(t) = \sum_\Gamma \frac{1}{\operatorname{Aut}(\Gamma)} t^\Gamma, \qquad t^\Gamma = t_2^{-E} \prod t_k^{V_k},$$

Review

- A vacuum fatgraph with V_k vertices of valence k has E edges, V vertices and $L = E - V$ loop order, with

$$L = 1 + \frac{1}{2}\sum_{k\geq 3}(k-2)V_k, \qquad E = \sum_{k\geq 3}\frac{k}{2}V_k, \qquad V = \sum_{k\geq 3}V_k$$

- For example, in the case of the *cubic* potential, $V(M) = t_2 M^2/2x + t_3 M^3/3$, the partition function Z is a generating function for counting cubic fat graphs, and

$$F = \sum_{g,p\geq 0} \hbar^{L-1} N^p \, t_3^V \, t_2^{-E} \, C_{g,p},$$

Outline

Virasoro Constraints

Varying $M \to M + \epsilon M^{n+1}$

$$Z = \int_{U(N)} dU \left(\prod_{i=1}^{N} d\lambda_i \right) \Delta(\lambda)^2 e^{-\frac{1}{\hbar}\sum_i V(\lambda_i)}, \qquad \Delta(\lambda) = \prod_{i<j} (\lambda_i - \lambda_j)$$

$$\delta \log d\lambda = \epsilon(k+1) \sum_i \lambda_i^k d\lambda.$$

$$\delta \log \Delta(\lambda)^2 = 2 \sum_{i<j} \frac{\lambda_i^{k+1} - \lambda_j^{k+1}}{\lambda_i - \lambda_j} = \epsilon \left(\sum_{i,j} \sum_{r=0}^{k} \lambda_i^r \lambda_j^{k-r} - (k+1) \sum_i \lambda_i^k \right),$$

$$\delta \left(e^{-\frac{1}{\hbar}\sum_i V(\lambda_i)} \right) = -\frac{\epsilon}{\hbar} \sum_i V'(\lambda_i) \lambda_i^{k+1} \left(e^{-\frac{1}{\hbar}\sum_i V(\lambda_i)} \right).$$

Schwinger Dyson Equation and Virasoro Constraints

- Recalling that $\operatorname{Tr} M^k = \sum \lambda_i^k$, we see that the Schwinger-Dyson equation for the Hermitian matrix model is
$$\left\langle \sum_{a+b=k}^{n} \operatorname{Tr}(M^a) \operatorname{Tr}\left(M^b\right) - \frac{1}{\hbar} \operatorname{Tr}\left(V'(M) M^{k+1}\right) \right\rangle = 0,$$

- where $\langle \cdots \rangle$ denotes the integral with respect to the measure $dM \exp(-\operatorname{Tr} V(m)/\hbar)$. Using
$$\frac{-\hbar}{Z} \frac{\partial Z}{\partial t_k} = \left\langle \operatorname{Tr} M^k \right\rangle, \qquad \frac{\hbar^2}{Z} \frac{\partial^2 Z}{\partial t_a \partial t_b} = \left\langle \operatorname{Tr} M^a \operatorname{Tr} M^b \right\rangle,$$

- the Schwinger-Dyson equations can be recast in the form $L_k Z = 0$ where
$$L_k = \hbar^2 \sum_{a+b=k} \partial_{t_a} \partial_{t_b} + \sum_i t_i \partial_{t_{k+i}}$$

Berends Giele

- Writing $Z = \exp(F)$ and using the perturbative expansion of the free energy as a series in $\hbar$ and the t_k's, the Virasoro constraints become

$$\sum_{a+b=k}\left(\sum_{\substack{g_1+g_2=g\\ p_1+p_2=p}} \frac{\partial G_{g_1,p_1}}{\partial t_a}\frac{\partial G_{g_2,p_2}}{\partial t_b} + \frac{\partial^2 G_{g-1,p}}{\partial t_a \partial t_b}\right) + \sum_i t_i \frac{\partial G_{g,p}}{\partial t_{k+i}} = 0$$

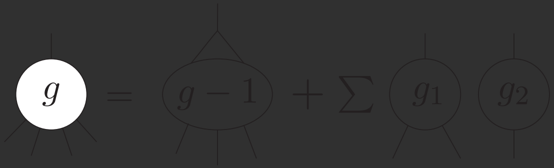

Figure: The loop equations schematically take the form of a "Berends-Giele" recursion labelled by genus g surfaces with p holes.

Adding Back External Legs

- One way to introduce 'external legs' is to evaluate correlation functions such as $\langle \mathrm{Tr}(M^n) \rangle$
- Just like the free energy F, we can expand these correlation functions in $\hbar$ and the 't Hooft coupling $\lambda \equiv \hbar N$

$$\langle \mathrm{Tr}(M^n) \rangle = \sum_{g,p} \hbar^{L-1+n} N^{p+n} \hat{G}_{g,p,1}(n),$$

- functions $\hat{G}_{g,p,1}(n)$ of the t_k, given by a sum over fatgraphs with: one trace factor that has n external legs; p closed loop boundaries (each contributing N); and that can be embedded in a genus g surface with p holes and 1 boundary

Adding Back External Legs

- Note that each external line of the fatgraph contributes $1/t_2$: in other words, we do not "amputate" the external propagators. Equivalently, we are studying vacuum fatgraphs that have one distinguished n-valent vertex
- For example, taking V to be the cubic potential, and keeping only connected diagrams,

$$\langle \mathrm{Tr}\big(M^2\big)\rangle_c = \hbar^{-2}\lambda^2(t_2)^{-1} + 3\hbar^{-2}\lambda^3(t_2)^{-4}t_3^2 + \dots,$$

- where the first term is a single "tree level" propagator diagram, and the second term are the planar 1-loop 2-point diagrams.

Resolvents

- Correlators can be organised into generating series called *resolvents*. For the single trace correlators, write

$$W = \lambda \left\langle \mathrm{Tr}\, \frac{1}{z-M} \right\rangle_c = \sum_{n=0}^{\infty} \omega_n z^{-n-1}, \qquad \omega_n = \lambda \langle \mathrm{Tr}\, M^n \rangle.$$

- Then the Virasoro constraints may be expressed in terms of W as

$$W(z)^2 + \frac{\hbar}{\lambda} W'(z) = \frac{1}{\lambda} V'(z) W(z) - R(z), \tag{1}$$

- Write W_0 for the $g = 0$ contribution to this sum, which corresponds to order $\hbar^{-2}$ (with λ fixed). Then the genus zero contributions are constrained by

$$\lambda W_0(z)^2 = V'(z) W_0(z) + \cdots .$$

Resolvents

- Picking out terms at fixed order z^{-n-2}, for $n > 0$, gives
$$\sum_{a+b=n} \omega_a^{(0)} \omega_b^{(0)} = \omega_{n+2}^{(0)} + \omega_{n+3}^{(0)}.$$
- In the λ expansion of the ω, the leading term is
$$\omega_n^{(0)} = \lambda^{n+1} \hat{G}(n) + \cdots ,$$
- where $\hat{G}(n)$ is the contribution from the n-point tree diagrams. So we see that
$$\hat{G}(n+2) = \sum_{a+b=n} \hat{G}(a)\hat{G}(b),$$

which is the usual "Berends-Giele" recursion for the Catalan numbers. The initial conditions on the recursion are $\hat{G}(1) = 0$, $\hat{G}(2) = 1$, $\hat{G}(3) = 1$.

Calculations using the Cut Equation

	L	g	h	$n=1$	2	3	4	5	
	0	0	1	x	x	1	2	5	
	1	0	1	1	3	10	35	126	
	1	0	2	x	1	4	(18,15)	(72,56)	(
	2	0	1	8	48	240	1120	5040	
	2	0	3	x	x	32	192		
	2	0	2	x	16	92			
	2	1	1	1	10				

Figure: The numbers of cubic diagrams computed by the cut equation.

Calculations using the Cut Equation

L	$n=1$	2	3	4	5	6
0	x	x	x	2	5	14
1	1	3	10	35	126	462
2	9	58				

Figure: Loop order expansion of $\langle \mathrm{Tr}(M^n) \rangle$ in the planar limit, computed by the cut equation.

Outline

The Stringy Integral

The Stringy integral

- Here, we regard them as α' deformations of surface functions, and use them as a toy model to study the most basic combinatorial content of open string integrals. In particular,
- At tree level this is just the statement that the field theory limit of the Veneziano amplitudes are tree amplitudes.
- For example, the surface function for the 4-point disk, $G_4 = x_{13} + x_{24}$ is the $\alpha' \to 0$ limit of the Euler beta function,

$$G_4^{\alpha'} = \int_0^\infty \frac{dy}{y} \left(\frac{y}{1+y}\right)^{\alpha'/x_{13}} \left(\frac{1}{1+y}\right)^{\alpha'/x_{24}} .$$

Laplace Transforms of $V_{g,n}$

$$G^{\alpha'}_{\bigcirc} = \int_0^\infty \frac{d\log y_1 d\log y_2 d\log y_3}{\text{MCG}} (y_1 y_2 y_3)^{\frac{\alpha'}{2x}}.$$

To evaluate this integral, it is useful to rewrite the measure as

$$G^{\alpha'}_{\bigcirc} = \int_0^\infty \frac{dL}{2}\, e^{-\frac{\alpha}{2x}L} \int\limits_{\mathcal{M}_{1,1}(L)} \mu_{WP},$$

where $L/2 = -\log(y_1 y_2 y_3)$ is the length of the boundary, and μ_{WP} is the Weil-Peterson volume form, restricted to the moduli space where the boundary has constant length L. This is given in our coordinates as

$$\mu_{WP} = \frac{dy_1 dy_2}{y_1 y_2} = \frac{dy_2 dy_3}{y_2 y_3} = \frac{dy_3 dy_1}{y_3 y_1}.$$

Laplace Transforms of $V_{g,n}$

When rewritten in different coordinates (Fenchel-Nielsen coordinates), the integral of μ_{WP} becomes identical to Mirzakhani's calculation of Weil-Peterson volumes using the McShane identity. This gives

$$V_{1,1}(L) \equiv \int_{\mathcal{M}_{1,1}(L)} \mu_{WP} = \frac{L^2}{24} + \frac{\pi^2}{6}.$$

So

$$G^{\alpha'}_{\bigcirc} = \frac{1}{3}\left(\frac{x}{\alpha'}\right)^3 + \zeta_2\left(\frac{x}{\alpha'}\right).$$

In particular, at leading order in α',

$$(\alpha')^3 G^{\alpha'}_{\bigcirc} \to G_{\bigcirc} = x^3/3.$$

α' Deformations

$G_S^{\alpha'}$ become polynomials in the matrix model limit, they cannot be realized as contributions to a one-matrix model with α' corrections. Take a potential $V(M) = \sum t_k M^k/k$, which is the cubic potential to leading order in α'. Then we add α' corrections to V:

$$t_2 = \frac{\alpha'}{x} + O((\alpha')^2), \qquad t_3 = 1 + O(\alpha'), \qquad t_k = O(\alpha') \ \ (k > 3).$$

The torus with a hole contributes at order $\hbar N$ to the free energy $F = \log Z$ of this model, with two diagrams: a $g = 1$ fatgraph with three edges and two cubic vertices, and a $g = 1$ fatgraph with two edges and one quartic vertex.

$$[\hbar N]\ F = t_3^2 t_2^{-3}\frac{1}{3} + t_4 t_2^{-2}\frac{1}{2}.$$

We recover $G_{1,1}^{\alpha'}$ if we set, for example, $t_2 = \alpha'/x$, $t_3 = 1 + \pi t_2/\sqrt{2}$, and $t_4 = -\sqrt{2}\pi t_2/3$. On the other hand, the sphere with three holes contributes at order $\hbar N^3$, also with two diagrams:

Comments and Questions!

1. Surfaceology
2. The Cut Equation
3. Matrix Models
4. Virasoro Constraints
5. The Stringy Integral