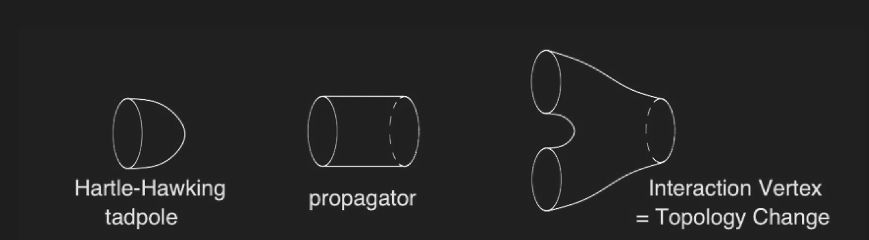
Liouville Theory as a 2D Bulk Quantum Gravity Theory and Matrix Models

Rahul K Balaji [1] Supervisor: Olga Papadoulaki [2]

[1]University of California Los Angeles [2]Perimeter Institute for Theoretical Physics

Overview and Motivation

- The aim of this program is to study the case of c=1 Liouville Theory having a dual description in terms of Matrix Quantum Mechanics (MQM) of N-ZZ D0 Branes. Here, instead of the conventional approach, where one interprets Liouville Theory as a worldsheet Conformal Field Theory (CFT, String Theory) embedded in a 2-dimensional target space, we take Liouville Theory as the Quantum Gravity Theory in bulk spacetime.
- This approach is corroborated by the fact that a holographic connection can be seen as in the case of a single Hermitian matrix model describing (2,p) minimal models coupled to gravity, where the physics of JT-gravity can be reached as a $p \to \infty$ limit of these models. We study the aforementioned theory since it is a richer UV-Complete theory of 2D-gravity with matter.
- The Matrix Models here do not play the role of their boundary duals, but give a direct link to the third quantized Hilbert Space description, i.e The target space of c=1 string plays the role of the superspace in which these two dimensional geometries are embedded.
- From the Matrix Model point of view, we introduce appropriate loop operators to create macroscopic boundaries on the bulk geometry. We do this in such a way that the boundary is of fixed size l and is related to the temperature β of the holographic dual theory.
- Here we are currently looking at two point macroscopic loop operator correlators corresponding to Euclidean wormhole geometry and three point correlators with a (local) Vertex operator on the same Geometry, which corresponds to the insertion of an operator on the boundary. We initially look at these objects at genus zero and then use MQM to study them at higher genera.



Liouville Theory And Equations of Motion

The Liouville Theory for the bulk spacetime with a boundary is defined by the action functional:

$$S = \int_{\mathcal{M}} d^2z\sqrt{g}\left(\frac{1}{4\pi}g^{ab}\partial_a\phi\partial_b\phi + \frac{1}{4\pi}QR\phi + \mu e^{2b\phi}\right) + \int_{\partial\mathcal{M}} du\, g^{1/4}\left(\frac{QK\phi}{2\pi} + \mu_B e^{b\phi}\right)$$

- ϕ is the Liouville field, and $(z,\bar{z})$ are the Euclidean target space coordinates.
- K is the extrinsic curvature, and Q is a parameter set by the underlying CFT. As for the case of $c=1$ here, $b=1$ and $Q=2$.
- μ_B is the boundary cosmological constant and μ is the bulk cosmological constant.
- A matter field $X(z,\bar{z})$ is also present.

Wheeler-DeWitt Perspective and Genus Zero Wavefunctions

After canonically quantizing the theory, and proceeding to study the spectrum of the Hilbert space of the theory, we arrive at the bulk **minisuperspace** Wheeler-Dewitt Equation:

$$\left(-\frac{\partial^2}{\partial\phi_0^2} + 4\mu e^{2\phi_0} - q^2\right)\Psi_q(\phi_0) = 0$$

- In the above equation q is the momentum conjugate to the zero mode of the matter field $X(z,\bar{z})$
- In case the surface has a boundary of fixed length l, then this can be expressed in terms of the zero mode as $l = e^{\phi_0}$, which is kept fixed.
- The wavefunctions correspnding to loops with macroscopic sizes at genus zero are:

$$\Psi_q^{macro}(l) = \frac{1}{\pi}\sqrt{q\sinh(\pi q)}K_{iq}(2\sqrt{\mu}l)$$

- These wavefunctions are exponentially damped for $l \to \infty$ and oscillate an infinite number of times for $l \to 0$.
- On the other hand, the microscopic states correspond to wavefunctions that diverge as $l \to 0$ and vanish as $l \to \infty$, and are given by the analytic continuation of the previous solution.

Matrix Quantum Mechanics and Loop Operators

- We now pass to the double scaled free fermionic theory description of the model, that would allows us to resum geometires corresponding to higher genera, and compute various observables.
- After diagonalizing the variables of matrix quantum mechanics and passing to the double scaling limit, we have the dynamics described by the second quantized non-relativistic fermionic field action:

$$S = \int dtd\lambda\hat{\psi}^\dagger(t,\lambda)\left(i\frac{\partial}{\partial t} + \frac{\partial^2}{\partial\lambda^2} + \frac{\lambda^2}{4}\right)\hat{\psi}(t,\lambda)$$

where, the double scaled fermi fields are defined using the normalized even/odd cylinder functions $\psi^s(\omega,\lambda)s = \pm$

$$\hat{\psi} = \int d\omega e^{i\omega t}\hat{b_s}(\omega)\psi^2(\omega,\lambda)$$

And the fermi-sea vacuum $|\mu\rangle$ (μ being the chemical potential), is defined by:

$$\hat{b_s}(\omega)|\mu\rangle = 0, \omega < \mu$$
$$\hat{b_s^\dagger}(\omega)|\mu\rangle = 0, \omega > \mu$$
$$\{\hat{b_s^\dagger}(\omega), \hat{b'_s}(\omega')\} = \delta_{s,s'}(\omega - \omega')$$

- We then define the fixed length matrix loop operator in terms of L, where L is the discrete lattice variable, and μ_B is the dual variable (which is the chemical potential) :

$$\hat{W}(L,x) = \frac{1}{N}e^{L\hat{M}(x)}$$
$$\hat{W}(\mu_B,x) = 1\frac{1}{N}\log[\mu_B - \hat{M}(x)] = \frac{1}{N}\sum_{l=1}^{\infty}\frac{1}{l}[\hat{M}(x)/\mu_B]^l - \log\mu_B$$

- We have the $1/N$ factor here to account for the symmetries in the Feynman diagram.
- In terms of the fermions, the density operator is:

$$\hat{\rho}(x,\lambda) = \hat{\psi}^\dagger(x,\lambda)\hat{\psi}(x,\lambda)$$

- Using this in the continuum limit, and taking the fourier transform of the macroscopic loop operator, we have

$$\hat{\mathcal{W}}(z,x) = \int_{-\infty}^{\infty} d\lambda\hat{\psi}^\dagger(x,\lambda)e^{iz}\hat{\psi}(x,\lambda)$$

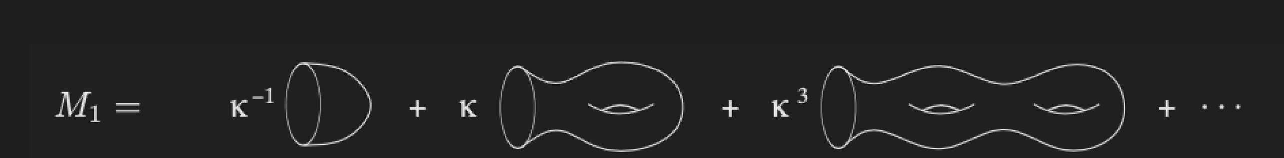
- And then we Wick rotate $z = \pm il$. The expectation value of the loop operator can be denoted as:

$$M_1(z,x) = \langle\hat{\psi}^\dagger e^{iz\lambda}\hat{\psi}\rangle$$

(Higher point Correlators are denoted by $M_n(z_i,x_i)$)

Loop Equations, Correlators, and Geometries - Hartle-Hawking Tadpole

In the case of the one Macroscopic loop, we have the Hartle-Hawking wavefunction:



Analytically, this is given by:

$$\Psi_{WdW}(l,\mu) = M_1(z=il,\mu) = \Re\left(i\int_0^\infty\frac{d\zeta}{\zeta}e^{i\zeta}\frac{e^{i\coth(\zeta/2\mu)\frac{z^2}{2}}}{\sinh(\zeta/2\mu)}\right)$$

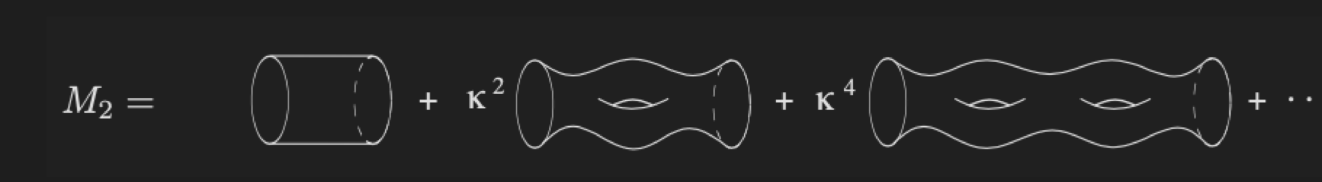
The genus expansion is recovered by restoring string coupling κ, according to: $\mu \to \mu/\kappa, l \to \kappa^{1/2}l, \lambda \to \kappa^{1/2}\lambda$

- The genus zero result is obtained from the previous equation by performing a $1/\mu$ expansion of the integrand:

$$\Psi_{WdW}^{(0)}(l,\mu) = \Re\left(2i\int_o^\infty\frac{d\zeta}{\zeta^2}e^{i\mu\zeta}e^{-i\frac{l^2}{\zeta}}\right) = \frac{2\sqrt{\mu}}{l}K_1(2\sqrt{\mu}l)$$

Loop Equations, Correlators, and Geometries - Propagator

- In the case of the two macroscopic loops, we have the correlator of the propagator, corresponding to the geometry of a Euclidean wormhole:



- In the two loop case, we have the following expression for the derivative of the correlator $\partial M_2(z_1,q,z_2,-q)/\partial\mu$

$$\Im\int_0^\infty\frac{d\zeta}{\sinh(\zeta/2)}\int_0^\infty dse^{-|q|s}\left(e^{iz_1z_2\frac{\cosh(s-\zeta/2)}{\sinh(\zeta/2)}} - e^{iz_1z_2\frac{\cosh(s+\zeta/2)}{\sinh(\zeta/2)}}\right)$$

- At genus zero, this expression is given by:

$$M_2(l_1,q,l_2,-q) = \int_{-\infty}^{\infty}dp\frac{1}{q^2+p^2}\frac{p}{sinh(\pi p}\Psi_p^{(macro)}(l_1)\Psi_p^{(macro)}(l_2)$$

Modified Wheeler Dewitt Equation for Higher Genera

- Considering the Hartle-Hawking tadpole case again, we can compute the derivative of M_1 with respect to μ exactly in terms of Whittaker functions (for q = 0). This has the interpretation as the one point function of the area operator $\langle\int e^{2\phi}\rangle$

$$\frac{\partial M_1(z,\mu)}{\partial\mu} = -\Re\left((-iz^2)^{-1/2}\Gamma\left(\frac{1}{2} - i\mu\right)W_{i\mu,0}(iz^2)\right)$$

- The Whittaker functions that appear in these expressions obey the equation:

$$\left(-\left[l\frac{\partial}{\partial l}\right]^2 + 4\mu l^2 + 4\eta^2 - l^4\right)\frac{W_{i\mu,0}}{l} = 0$$

This is the generalization of the WdW equation that we found from Liouville theory to include higher genera.

Discussion and Future Directions

We have shown how matrix model technology can be used to describe Liouville Theory in the bulk, in a third quantized picture. The future directions in which this program can be extended has to do with computing the expectation values of insertion of local vertex operators. This is achieved by considering 3 macroscopic loops initially (corresponding to the pants topology) and shrinking one of the loops into being a vertex operator.

Matrix Model Dictionary and References

Quantum Gravity	Matrix Model	Boundary Dual
Liouville potential $\mu e^{2\phi}$	Inverted osciallator potential	-
Cosmological constant μ	Chemical potential $-\mu$	IR mass gap μ
D0 particle (ϕ: D, X: N)	Matrix eigenvalue λ	Energy eigenvalue E_i
Boundary: $S_{bdy} = \mu_B\oint e^{\phi}$	Loop operator: $\langle\log[z-\lambda]\rangle$	Microcanonical $\langle\rho_{dual}(E)\rangle$
Boundary cosmological constant μ_B	Loop parameter z	Energy E
Fixed size body $l = e^{\phi_0}$	Loop length l	Inverse temperature β
WdW wavefunction $\Psi(l)$	Fixed size loop operator $\langle M_1(l)\rangle$	Partition function $Z_{dual}(\beta)$
Third quantized vacuum	Fermi sea of eigenvalues	-
S-matrix of universes	S-matrix of density quanta	-
Two boundaries: $l_{1,2}$	Loop correlator $\langle M_2(l_1,l_2)\rangle$	SFF:$l_{1,2} = \beta \pm it$
Two boundaries: μ_B	Density correlator $\langle\rho(_1)\rho(\lambda_2)$	Density of states correlator